Problem 11208, from Volume 133 No. 3 p. 267
Menger Sponge Map-Coloring

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To answer this question, we first find the genus of each M_n. Let P_n be the topological genus of the surface M_n, and let B_n be the number of holes on one face of M_n (one face in terms of the six original faces of M_0). Clearly $P_0 = 0$, $P_1 = 5$, and $B_n = \sum_{k=0}^{n-1} 8^k = \frac{8^n-1}{7}$. We will show that $P_n = 20P_{n-1} - 24B_{n-1} + 5 = \frac{3}{19}20^n + \frac{2}{7}8^n - \frac{59}{133}$.

M_n can be constructed by arranging 20 copies of M_{n-1}. Note that there are 24 places where the face of an M_{n-1} meets another face of an M_{n-1}. Counting holes as 20 copies of M_{n-1} double-counts the holes on such faces, giving $24B_{n-1}$ double-counted holes. When the 20 M_{n-1}'s assemble, they take the same shape as M_1, generating $P_1 = 5$ additional holes between them, giving $P_n = 20P_{n-1} - 24B_{n-1} + 5$. To eliminate the recursion here, we use generating functions and obtain $P_n = \frac{3}{19}20^n + \frac{2}{7}8^n - \frac{59}{133}$.

A result of G. Ringels and J. W. T. Young ["Solution of the Heawood Map-Coloring Problem." *Proc. Nat. Acad. Sci. USA 60, 438-445, 1968.*] tells us the chromatic number of a surface of genus $p > 0$ is Heawood's number $H = \lfloor \frac{7+\sqrt{1+48p}}{2} \rfloor$. Thus the chromatic number for any M_n is

$$H_n = \left\lfloor \frac{7 + \sqrt{1 + 48(\frac{3}{19}20^n + \frac{2}{7}8^n - \frac{59}{133})}}{2} \right\rfloor.$$

The first few values of H_n are 4, 11, 34, 133, and 566. It is interesting (although not entirely surprising) that H_n grows exponentially ($H_{100} > 10^{65}$).