Math 152 **Project 3: Due Dec. 5** **Fall 2017**

This document contains general instructions for completing this assignment, followed by some background leading up to the project, and then the prompt(s) for the project. The grading rubric is also included.

First, let us remember the purposes of the projects in this course: To foster the use of computer-assisted methods in solving mathematical and scientific problems; To engage in constructive, in-depth problem solving to explore course material; and to practice writing in a technical context.

Your paper should be formatted in a reasonable way. You should include your name, the name of the class, the semester, the assignment, and a title you choose for your paper. ("Math Paper" or "Joe Student" or "Paper 2" is not a title.)

You may type your paper using any software with which you are familiar. Examples of software you could use are MS Word + Equation Editor, LaTeX, MathType, or even Mathematica or Maple (both of which have word processing features). In any event, though, your paper should be typed. You will submit a printed copy of this and any other physical supporting materials.

You will also send a PDF version of this paper, the source document (DOC, DOCX, etc.), and any other electronic materials (e.g. code, image files, etc.) by email. These should be submitted the same day as the hard-copy, the due date. The PDF is important – if you only send the DOCX file, it may not display correctly. Please send both.

Your margins, spacing, font choice, etc. should be standard: Margins should be 1" give or take, font should be a standard font in 12 point, and lines should be 1.5-, or double-spaced. Figures and equations should be included appropriately when needed. If you cannot type all your equations, you can do it the old-fashioned way and leave a blank space, then write them in pen on your print-out.

Your paper should address directly, and in depth, the question in the prompt. You should explore the question/topic in detail. The creativity, thoroughness, and correctness of your response will comprise a major component of your grade. You are required not only to find "the answer," but to discuss the solution, how you found it, and what it says about the problem (which will vary by prompt). Although you are not required to include every possible type of topic we have covered, failing to include a relevant type of method, idea, concept, visual, etc. (especially one explicitly mentioned in the prompt) may result in a deduction if your answer seems incomplete.

For computer-based portions of the work, the instructions below and everything else we've covered so far in class should be enough to get you started. If there are any questions about this, please send me an email.

Your paper should be roughly 2 pages in length, or about 500 words. Your paper may be longer, especially if you double-space or use a large number of images. The absolute minimum is 400 words, which should span about 1.5-2 pages (assuming two graphics). Papers below this threshold may be subject to automatic deductions. Your paper should include the appropriate amount (and balance) of mathematics and exposition.

If you are concerned about the length of your paper, put length aside. It is most important that you address the question thoroughly – err on the side of thoroughness. That means make sure you answer the question as well and as completely as you can, not just that you add enough filler or rambling that it looks like you did.

Students may work individually or in groups of 2-3. Students should work together in a fair collaboration, with each student contributing equitably to each aspect of the paper. Students working in groups need only hand in one copy of the paper together.

The basic grading breakdown is as follows:

- Mathematics: Analytical, Qualitative, Numerical 30%
- Presentation: Visuals, Layout, Format, Professionalism 30%
- Writing: Exposition, Organization, Clarity, Usage 40%

The details are included in the rubric (below). This rubric page should be submitted as a cover page for your report.

NOTICE: This is a firm reminder (one that should be unnecessary) that academic honesty is required for this (and all) assignments. You should be referring to the ideas in our coursework and course materials to respond to your prompt. You may use third-party resources to *supplement* your work, but you should not look up solutions to these problems and derive your solutions by copying these. If you find interesting quotations or ideas, you must quote them properly and clearly, attributing them to your sources. If you find ideas from a source and use them in your paper, cite your source, even if it's not a quotation.

Besides working within a team, you are not permitted to collaborate with others or copy others' work. When plagiarism occurs, it is usually obvious, and it will land you in trouble. This should go without saying, but I have found that this reminder is sometimes useful.

Project 3 Background Material

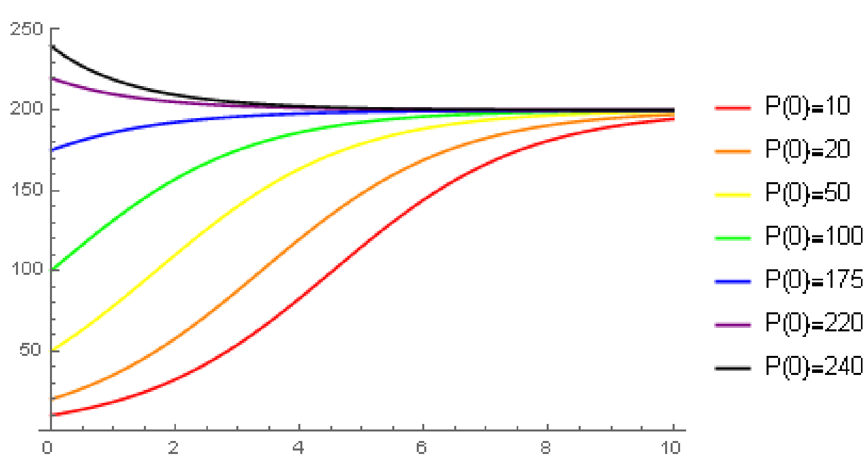
We have learned about functions and how they describe quantities in scientific problems. We will be learning about differential equations: relationships that arise in scientific contexts that relate a particular quantity to its own derivative.

In order to understand these quantities – or systems of multiple quantities – we study the equilibrium values of the quantities. For example, we have seen populations modeled by certain functions, and we discussed the long-term behavior of those functions. We discussed how such a function $P(t)$ has a limit: $\lim_{t\to\infty} P(t)$, which we might have called the stable population, the long-term population, or the carrying capacity of the environment in which the population lives.

When we study a system that is governed by one or more differential equations, we look for equilibrium states by setting the derivative(s) equal to zero – equilibrium means a system that is in balance, at rest, with no change occurring. In most (but not all) cases, we're studying systems in which the independent variable is time.

We also discussed how the starting value of the function (the "initial value" or $P(0)$) will determine the function $P(t)$ different. A population that starts near its carrying capacity will just grow incrementally towards that equilibrium value. A poplulation that starts very low might seem to grow exponentially before slowing its growth as it approaches the equilibrium. We have seen that integration produces a $+C$, and so does solving a differential equation. This $+C$ can be determined from the initial value, and so the "solution" to a differential equation (like the solution to an integral) is really a family of solutions, each corresponding to a different initial value. It is often very important to understand how the initial value relates to the long-term behavior or other characteristics of the function.

For example, below we have the graph of several functions of the form $P(t) = \frac{k}{1+Ce^{-rt}}$. This is a population growth model we studied in previous chapters because its limit is easy to compute: $\lim_{t\to\infty} P(t) = k$. However, we now know that various values of C correspond to certain initial values $P(0)$, as exmplified by the plots below.



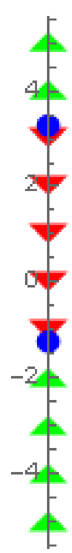
This is not possible in all cases. In general, we hope to understand how this system behaves, but if the environment or other factors in the system change independently, over time, there may be no equilibrium at all. Imagine a sunflower planted under a lamp that never moves.

The sunflower will sit, facing the lamp, never turning left or right, because the lamp does not vary over time. The sunflower is at equilibrium, in terms of which way it faces. However, in reality, a sunflower is never at rest – it turns to face the rising sun, faces up to the afternoon sun, and turns towards the sun as it sets. (This process is called heliotropism.)

In mathematical terms, the systems we study in this fashion must not be time-sensitive. That doesn't mean they aren't still functions of time, they certainly are. But what it means is that in general, we may express y' as a function of t and y. But in these cases, we need y' to be a function of only y. This is called "autonomous" – the variable y is, in some sense, "in charge of" itself (which is what the word autonomous means), because y is the only quantity that determines y'.

Take, for example, the differential equation (from now on, we'll just say "DE") $y' = f(y) = y^2 - 2y - 4$. We first go about finding the equilibrium values of y by setting $y' = 0$. The quadratic formula gives us two irrational values of y, precisely $1 \pm \sqrt{5}$, which I will call $y_1 \approx -1.23607$ and $y_2 \approx 3.2607$.

If $y(0)$ happens to be one of these values, the function will never change because in that case, y' is always zero. However, we might be interested in seeing which of these two solutions could correspond to the eventual outcome of solutions where $y(0)$ is not y_1 or y_2. As described in Chapter 26, we can analyze whether certain values of y will lead to an increase or decrease (i.e. whether $y' = f(y)$ is positive or negative). We use this to construct a chart indicating where the function is increasing or decreasing, with respect to y. We simply test $f(y)$ at values other than the y_1 and y_2 and determine whether it is positive or negative. We come up with this sort of image:



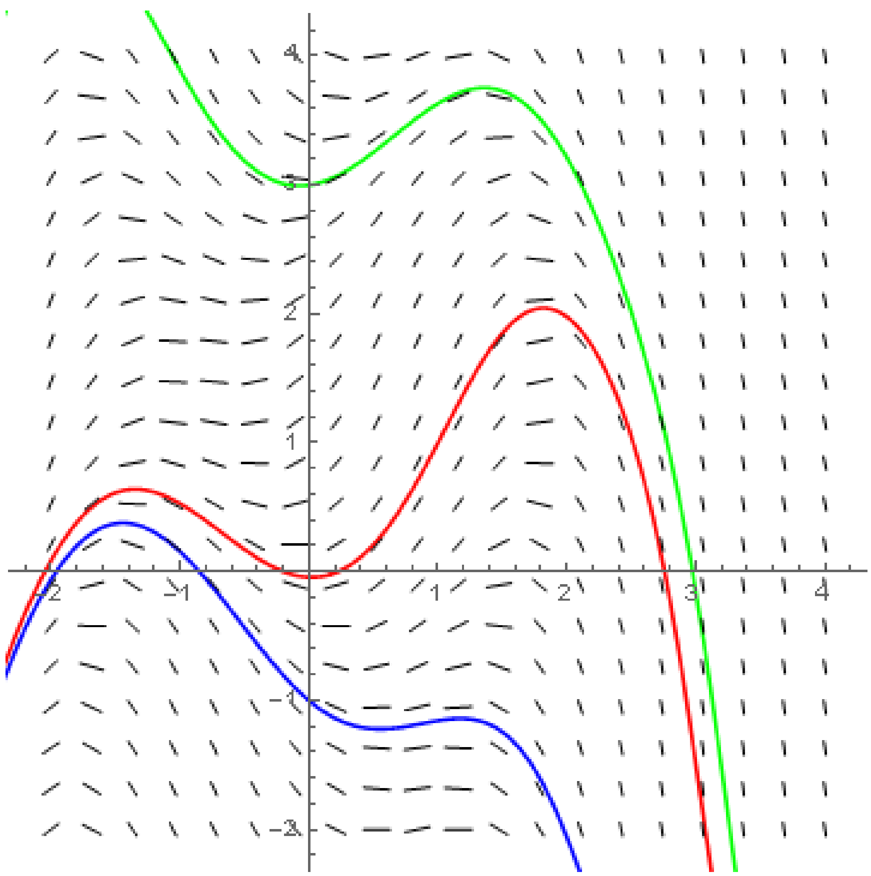
We can see that near y_2 (the higher of the two blue dots, which represent equilibria), arrows point away from the equilibrium. This means this equilibrium is unstable – in the long term, solutions to the DE will tend to move away from y_2. But for y_1, solutions seem to move towards this value. This value is stable, and would be the stable state for any solution whose $y(0)$ value puts it in range to move towards y_1. That range is any $y(0)$ less than y_2. Those solutions that start higher than y_2 will simply grow without bound.

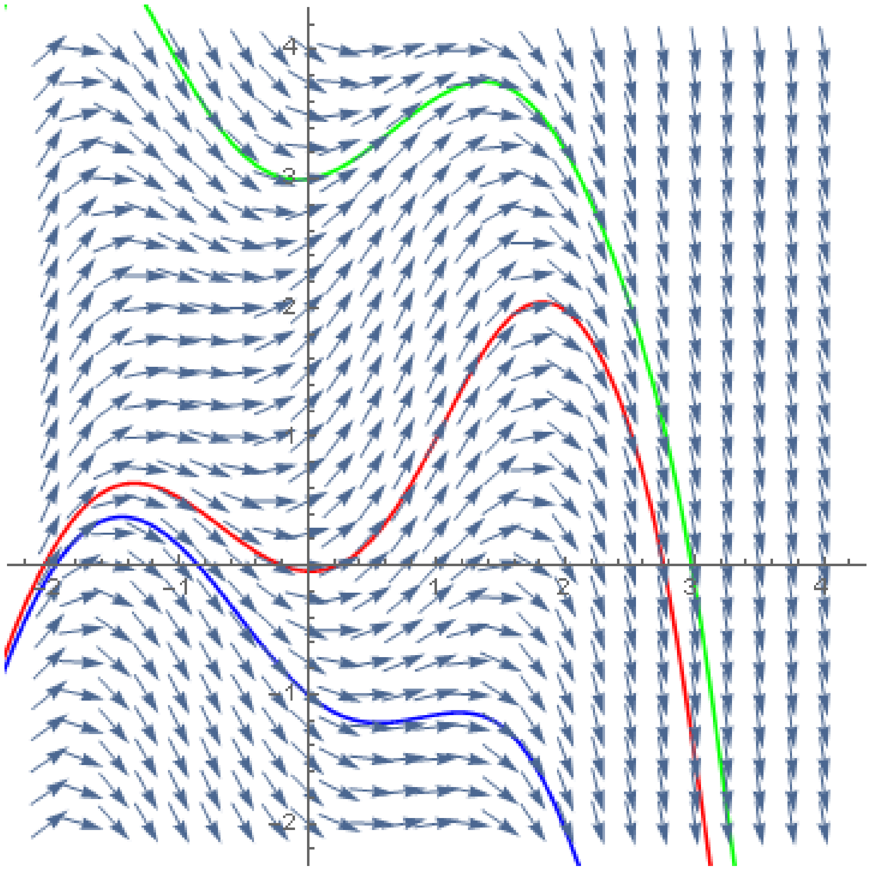
Notice that through this analysis, we did not solve the DE. We could develop methods to do this, but we are focused on this qualitative analysis: Where are the equilibrium solutions, and how do solutions behave with respect to those solutions?

We will discuss how to solve some DEs in the future, but we can also apply our qualitative analysis to systems of two (or more) DEs – and in this course, we will not discuss any other methods for analyzing systems of DEs.

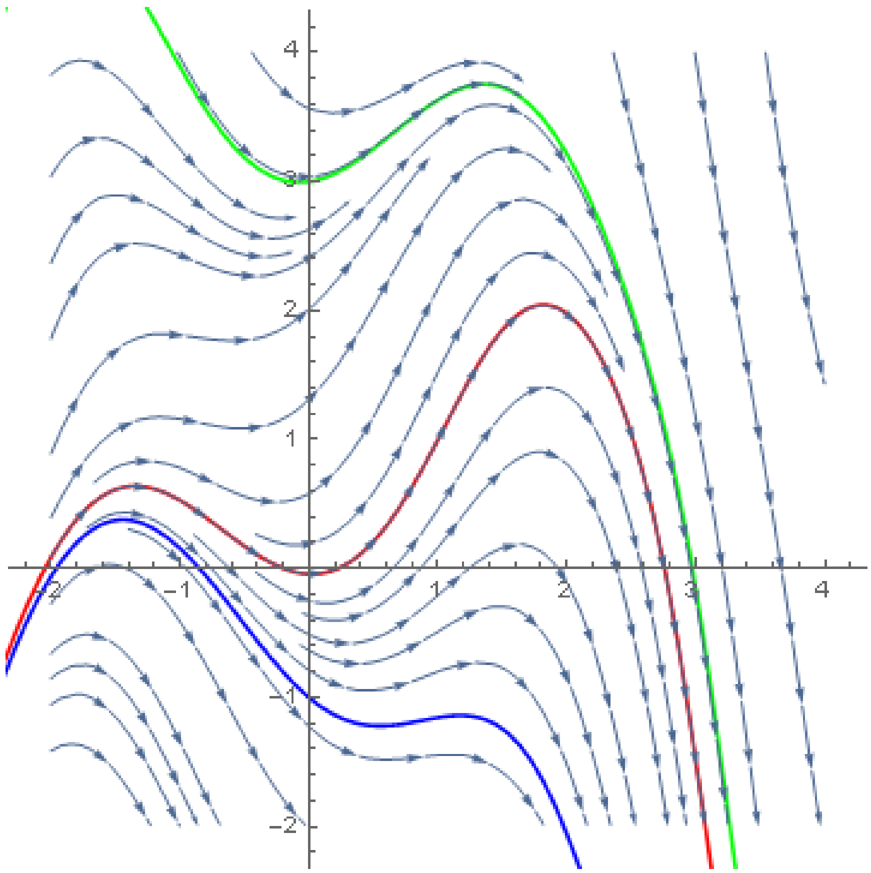
If we consider a system of two differential equations, also assuming that these equations are autonomous, we end up with two differential equations we could write as: $x' = f(x, y)$ and $y' = g(x, y)$, where x and y are each functions of t. And to analyze this, we could consider the fact that relationship between x and y can be determined by looking at these slopes: at each point in the $x - y$ plane, we can determine x' and y' and draw a line that reflects this directionality. Some authors prefer drawing arrows, making a "vector field" out of the image. Others will simply draw lines with the appropriate slope, making a "slope field" instead.

Solutions will follow these guiding lines because they must have the correct x' and y' values at every point they move through. See below for an example of each of these, along with some solution curves shown.





Notice that these fields and the solution curves do not tell you how long it takes for a particular solution to move in the $x - y$ plane, because t is not a variable in this illustration (it is only implicit in that x and y are both functions of t). The Wolfram language has a simpler way of doing this, which draws a number of different trajectories of possible solutions – a good way of combining the idea of drawing a slope/vector field and looking at solutions in that field. Instead, draw a field of solutions.



We can use these images to analyze the systems – equilibria can be found in these systems, algebraically or visually, and we can describe how solutions behave relative to those equilibrium points based on what we see in the plot. There are many mathematical techniques that take those observations further, but we will only go so far as to learn to be conversant – we want to describe what we see in the picture.

Project 3 Prompt

You work in a new special EPA taskforce as an analyst, studying the dynamics of a number of ecological, biological, and industrial systems related to EPA-regulated matters. Please assemble a group of 1-3 analysts and contact your EPA supervisor, Dr. Myers, who will assign you to a specific project.

You will be assigned two separate items to analyze. They may or may not be related. One item will be a single differential equation. You will need to identify the equilibrium values of the function, discuss the stability of those equilibria, and explain how solutions that originate at other values will tend towards those values.

The second item will be a system of two equations. You will be asked to produce a visual representation of the system (as above) and to describe in detail the behavior of the system. Locate the equilibrium point(s), and say whether they seem to be stable (solutions go towards them), unstable (solutions go away from them), or something else. Describe the regions of the image that go towards each stable point. Describe how solutions appear to move towards stable point(s) – straight line? spiral? zig-zag? You may locate the equilibrium points visually or using algebra to solve $f(x, y) = 0, g(x, y) = 0$. Use your discretion when you decide whether you can "eyeball" it, or whether you should use algebra instead.

For your report, please be sure to include some graphics. You can include the "sign chart" indicating the stability of your single DE. You can also use commands like `Plot` and `NDSolve` in Wolfram, or `ode45` and `plot` in MatLab, or `DEplot` in Maple, to sketch solutions to that system (without actually solving them explicitly). For more on these methods, see:

- `http://reference.wolfram.com/language/howto/PlotTheResultsOfNDSolve.html`
- `http://terpconnect.umd.edu/~petersd/246/matlabode.html#L588`
- `https://www.maplesoft.com/applications/view.aspx?SID=4104&view=html`

You should also include in your report an image for your 2D system of two DEs. To do this, you will need to use the command `StreamPLot` in Wolfram, or `dirfield` in MatLab, or `DEplot` in Maple. You are strongly encouraged to use Wolfram's "stream plot." It gives you a very accurate, easy-to-read visual compared to slope fields or other rough, potentially misleading visuals. For more on these methods, see:

- `https://reference.wolfram.com/language/ref/StreamPlot.html`
- `http://terpconnect.umd.edu/~petersd/246/matlabode.html#L585`
- `https://www.maplesoft.com/applications/view.aspx?SID=4701&view=html`

You should also, of course, use the computer to do computational tasks like determining values of y that make your function $f(y)$ equal to zero (same for $f(x, y), g(x, y)$), or plugging in values of y to test whether $f(y)$ is positive or negative.

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Name(s): ______________________________

Item	Subitem	Points	/	Total
Mathematics	Analytical		/	5
	Qualitative		/	15
	Numerical		/	5
	Other		/	5
	Total		/	30
Presentation	Visuals/Layout		/	15
	Format		/	10
	Professionalism		/	5
	Total		/	30
Writing	Exposition		/	15
	Organization		/	10
	Clarity		/	8
	Usage		/	7
	Total		/	40
	Deductions			
GRAND	TOTAL		/	100

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