MATH 435 In-Class Activity 4 – February 3 Spring 2020

Name: Solutions Section: 100

Please complete the following exercises. You may collaborate with your classmates, consult your notes or text, and/or ask for help. Note that participation in this activity is not optional.

1. Confirm that each function satisfies the Cauchy-Riemann equations.

(a) $h(z) = z + 2i$

$h(x+yi) = x+yi+2i = x+(y+2)i$

$u = x$ $\quad$ $v = y+2$ $\qquad$ $u_x = v_y$

$u_x = 1$ $\quad$ $v_x = 0$ $\qquad$ $v_x = -u_y$

$u_y = 0$ $\quad$ $v_y = 1$

(b) $f(z) = z^2 + z$

$f(x+yi) = (x+yi)^2 + x+yi$

$= x^2 + 2xyi - y^2 + x + yi$

$= (x^2 - y^2 + x) + (2xy+y)i$

$u = x^2 - y^2 + x$ $\quad$ $v = 2xy + y$

$u_x = 2x+1$ $\quad$ $v_x = 2y$

$u_y = -2y$ $\quad$ $v_y = 2x+1$

(c) $g(z) = \frac{1}{z}$ (except, of course, at $z = 0$)

$$g(x+yi) = \frac{1}{x+yi} = \frac{1}{x+yi}\cdot\frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2} = \frac{x}{x^2+y^2} + \left(-\frac{y}{x^2+y^2}\right)i$$

$$u = \frac{x}{x^2+y^2} \qquad v = -\frac{y}{x^2+y^2}$$

$$u_x = \frac{1\cdot(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{0\cdot(x^2+y^2) - 2y\cdot x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = -\frac{0\cdot(x^2+y^2) - 2x\cdot y}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$v_y = -\frac{1\cdot(x^2+y^2) - 2y\cdot y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2}$$

OVER

2. Consider following function (defined in terms of $z = x + iy$, where x and y are the real and imaginary parts of z, respectively).

$$f(x+iy) = \big(x^3 + 3x^2y - y^3 - x^2 - 2y^2\big) + \big(-x^3 + 3xy^2 - y^3 + 4xy + 3y\big)i$$

(a) Where does f satisfy CR? Describe the set of points clearly.

$u_x = 3x^2 + 6xy - 2x$ $\qquad$ $v_x = -3x^2 + 3y^2 + 4y$

$u_y = -v_x$ always

$u_y = 3x^2 - 3y^2 - 4y$ $\qquad$ $v_y = 6xy - 3y^2 + 4x + 3$

u_x is not equal to v_y unless:

$$3x^2 + \cancel{6xy} - 2x = \cancel{6xy} - 3y^2 + 4x + 3$$

$$3x^2 - 2x + 3y - 4x = 3$$

$$3x^2 - 6x + 3y^2 = 3$$

$$x^2 - 2x + y^2 = 1$$

$$x^2 - 2x + 1 + y^2 = 2$$

$$(x-1)^2 + y^2 = 2$$

This is a circle, centered at $(1,0)$ with radius $\sqrt{2}$.

(b) Is the function differentiable at those points?

<u>Yes</u> because our four partials (u_x u_y v_x and v_y) are continuous in a neighborhood of any point on the circle. (Indeed, they are continuous everywhere.)

(c) If yes, what is the derivative?

Since we have u_x u_y v_x v_y already, we can use either $f'(z) = u_x + i v_x$ or $v_y - i u_y$

$$f'(x+iy) = u_x + i v_y = 3x^2 + 6xy - 2x + (-3x^2 + 3y^2 + 4y)i$$

Example adapted from John Still, UNSW.